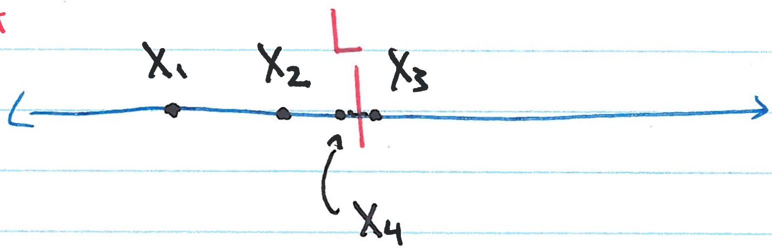
Announcements:
- HW pickup in MLC (LSK 300)
- Reminder: conflicts w/ Midterm (Feb 11, evening) 6:30 − 8:30

Last time:

If $x_1, x_2, x_3, \dots$ are real numbers, We say $x_1, x_2, \dots$ **converges** if

$$\exists L \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N \Rightarrow |x_n - L| < \varepsilon)$$

"limit"



Ex: $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots$ $x_n = \frac{1}{10^n}$

Prop: This sequence has limit $L = 0$.

Proof: We need to prove

"$\forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N \Rightarrow |x_n - 0| < \varepsilon)$"

Let's take ε to be any positive real number

We must show $\exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N \Rightarrow |x_n - 0| < \varepsilon)$

We will pick $N = \lceil \log_{10}(1/\varepsilon) \rceil$ ← round up to the nearest whole number.

We need to show that

$$\forall n \in \mathbb{N}, \left(n > \lceil \log_{10}(\tfrac{1}{\varepsilon}) \rceil \Rightarrow |x_n| < \varepsilon\right)$$

Suppose n is any natural number s.t. $n > \lceil \log_{10}(\frac{1}{\varepsilon}) \rceil$. Then

$$10^n > 10^{\lceil \log_{10}(\frac{1}{\varepsilon}) \rceil} > \frac{1}{\varepsilon}.$$

Therefore, $\varepsilon > \frac{1}{10^n}$. But $\frac{1}{10^n} = x_n$.

Therefore, we conclude $|x_n| < \varepsilon$.

Therefore, $N = \lceil \log_{10}(\frac{1}{\varepsilon}) \rceil$ works.

So we're proven the statement for any $\varepsilon \in \mathbb{R}_{>0}$. ■

$\frac{1}{10^n} < \varepsilon$

$\frac{1}{\varepsilon} < 10^n$

$\boxed{\log_{10}(\frac{1}{\varepsilon}) < n}$

<u>Exercise</u>: Prove that the sequence $x_n = 1 - \frac{1}{2^n}$ converges.

<u>Exercise</u>: Prove that the sequence $-1, 1, -1, 1, -1, 1, \ldots$ does <u>not</u> converge.

<u>Prop</u>: The sequence $x_n = 1 - \frac{1}{2^n}$ converges

<u>Pf</u>: We must show $\exists L \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N},$ $\forall n > N, \left|1 - \frac{1}{2^n} - L\right| < \varepsilon$

We will show $L = 1$ works.

↳ We must show $\forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}, \forall n > N, \underbrace{\left|1 - \frac{1}{2^n} - 1\right| < \varepsilon}$

Let ε be any positive real number.

$$\left|-\frac{1}{2^n}\right| < \varepsilon$$

$$\frac{1}{2^n} < \varepsilon$$

↳ We must show $\exists N \in \mathbb{N}, \forall n > N, \frac{1}{2^n} < \varepsilon$.

We will show $N = \lceil \log_2(1/\varepsilon) \rceil$ works.

↳ We must show $\forall n > \lceil \log_2(1/\varepsilon) \rceil, \frac{1}{2^n} < \varepsilon$.

If $n > \lceil \log_2(1/\varepsilon) \rceil$, then $2^n > 1/\varepsilon$.

So $\frac{1}{2^n} < \varepsilon$, as desired.

We have shown $N = \lceil \log_2(1/\varepsilon) \rceil$ works.

We have shown the required statement for every ε.

We have shown $L = 1$ works. ∎

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<u>Prop</u>: The sequence $-1, 1, -1, 1, \dots$ does not converge.

<u>Pf</u>: We must show $\sim \exists L \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}, \forall n > N, |x_n - L| < \varepsilon$

Which is the same as

$$\forall L \in \mathbb{R}, \exists \varepsilon \in \mathbb{R}_{>0}, \forall N \in \mathbb{N}, \exists n > N, |x_n - L| \geq \varepsilon.$$

Here, $x_n = (-1)^n$.

Let L be any real number.

↳ We claim $\varepsilon = 1$ satisfies the condition

↳ Let N be any positive integer

↳ We must find some ~~number~~ integer $n > N$ s.t. $|(-1)^n - L| \geq 1$.

- If $L \geq 0$ and N is even, take $n = N+1$.
  Then $(-1)^n = -1$, and so $|(-1)^n - L| \geq |-1 - 0| = 1$.
- If $L \geq 0$ and N is odd, take $n = N+2$.
  Then $(-1)^n = -1$, and so $|(-1)^n - L| \geq |-1 - 0| = 1$
- If $L \leq 0$ and N is even, take $n = N+2$.
  Then $(-1)^n = 1$, so $|(-1)^n - L| \geq |1 - 0| = 1$
- If $L \leq 0$ and N is odd, take $n = N+1$.
  Then $(-1)^n = 1$, so $|(-1)^n - L| \geq |1 - 0| = 1$.

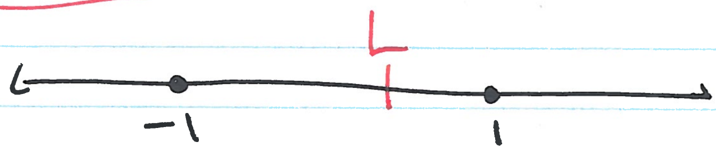
In all four cases, we have constucted an n which works.

We have shown that any N works.

We have shown that $\varepsilon = 1$ works.

We have shown that any L works. ∎

Intuition for ⑤:



I want to show that no matter what L is, there are infinitely many ~~elements~~ numbers in the sequence that are not close to L.

- If $L \geq 0$, then the -1's are far away.
- If $L \leq 0$, then the $+1$'s are far away.